

-Trek:*The One-Loop $\mathcal{N} = 4$ Noncommutative SYM Action**

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ABSTRACT: We investigate $\mathcal{N} = 4$ noncommutative super Yang-Mills (SYM) theory. We compute the one-loop four gauge boson scattering amplitude on parallel D_p-branes, and find the corresponding contribution to the noncommutative SYM one-loop action in a momentum expansion. The result is somewhat surprising. We find that while the planar diagram can be written using the usual *-product, the contributions from non-planar diagrams in general involve additional structure beyond the *-product, arising from the nontrivial worldsheet correlations surviving the field theory limit. To each nonplanar diagram, depending on the number n of external vertex operator insertions on each boundary, there is a corresponding $*_n$ n -ary operation. We further find that it is no longer possible to write down an off-shell gauge invariant one-loop effective action using the noncommutative field strength defined at tree-level.

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1. Introduction

Field and string theories on noncommutative space(time) have attracted much attention recently [1, 2]. They appear naturally in various decoupling limits of the worldvolume theories of D-branes in a background NS-NS B -field [3, 4, 5, 6, 7, 8]. While much progress has been made in understanding the perturbative dynamics [9, 10, 11, 12] and their strong coupling limits [4, 5, 13, 6, 8], many fundamental issues remain obscure.

In noncommutative gauge theories, gauge invariance becomes much more subtle. The gauge group and the allowed representations of the gauge group are highly constrained. A basic example is that in noncommutative field theory, $SO(2)$ is not a good gauge group even though the isomorphic $U(1)$ is. The first statement follows from the fact that

$$\begin{aligned} \begin{pmatrix} 0 & f(x) \\ -f(x) & 0 \end{pmatrix} * \begin{pmatrix} 0 & g(x) \\ -g(x) & 0 \end{pmatrix} - \begin{pmatrix} 0 & g(x) \\ -g(x) & 0 \end{pmatrix} * \begin{pmatrix} 0 & f(x) \\ -f(x) & 0 \end{pmatrix} \\ = \begin{pmatrix} -f(x)*g(x)+g(x)*f(x) & 0 \\ 0 & -f(x)*g(x)+g(x)*f(x) \end{pmatrix}, \quad (1.1) \end{aligned}$$

which is not an element of the $SO(2)$ algebra. It is also interesting that, in theories with only adjoint matter, translations along noncommutative directions are a subset of gauge transformations, and thus there are no gauge invariant local operators [14, 15].¹

Another puzzling issue is the contributions of the nonplanar diagrams to the 1PI effective action of the noncommutative field theories. One generically encounters IR singularities when an external momentum crosses an internal line in a nonplanar diagram for which the internal momentum integration is UV divergent at $\Theta = 0$; this is the so-called IR/UV mixing. The presence of IR/UV mixing indicates that the long distance behaviour of the system is no longer insensitive to the short distance physics, and the UV cutoff might not be pushed to infinity in a consistent way (see, however, [17]).

In [18], it was suggested that IR/UV mixing may be naturally understood in terms of a “stretched string”. When $B \neq 0$, an open string has a nonzero stretched displacement $\Delta x = \Theta k$,² even in the field theory limit $\alpha' \rightarrow 0$. As the momentum k is increased, the string stretches longer. Thus when there is a net momentum flow k between the two ends of the open string, as in the case of a nonplanar diagram, the integrated loop particle has to propagate an additional distance $|\Delta x|$, the effect of which is to seemingly cut open an otherwise closed first-quantized particle loop. When, for $B = 0$, the loop integral is divergent at short distances, the stretched string acts as an effective short-distance cutoff and regularizes the divergence. However, since taking Δx to zero (IR for external momentum k) will recover the original UV divergence, the amplitude is generally singular as $\Delta x \rightarrow 0$. Thus, $k \rightarrow 0$ is not a continuous limit, and the singularity at small k reflects the UV divergence of the $B = 0$ theory. The stretched string can also be used to understand two-loop computations.[22]

In this paper we further explore the IR/UV effects of the stretched string. To keep the IR singularities from obscuring our understanding, we consider a highly supersymmetric example: $\mathcal{N} = 4$ noncommutative SYM theory. Since this theory is finite in the commutative limit, the nonplanar diagrams are not plagued by the IR singularities mentioned above. We are thus able to give an explicit and unambiguous result for its one-loop scattering amplitudes. Various issues, including the contributions from

¹Matter in other representations can sometimes be used to construct gauge invariant local operators.[16]

² k is the center of mass momentum of the string. Semi-classically, the length Δx may be understood via the dipole picture discussed in [19, 20, 21].

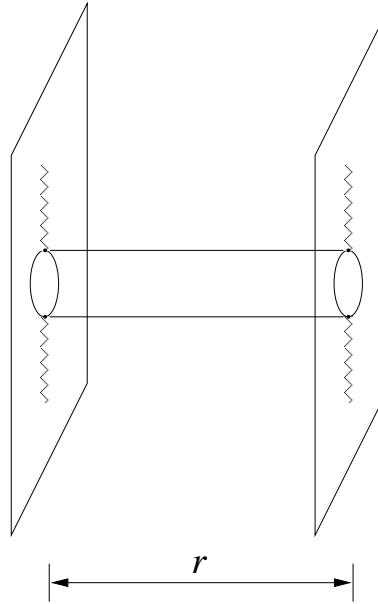


Figure 1: An example of the scattering amplitude considered. This is the nonplanar diagram with two gluons on each brane (a $2 \rightarrow 2$ process).

nonplanar diagrams and one-loop off-shell gauge invariance can then be discussed in a controlled manner.

Like the commutative theory, there is no 1-loop contribution to the two and three-point amplitudes in $\mathcal{N} = 4$ noncommutative SYM theory. We will thus focus on the four-point amplitudes, specifically the four gauge boson annulus amplitude (*e.g.* figure 1). The leading order contribution in the low energy limit determines the F^4 term in the low energy effective action of noncommutative $\mathcal{N} = 4$ super-Yang-Mills theory. Since we are interested in exploring the effect of noncommutativity, we take the low energy limit to mean³

$$\frac{k_i G^{-1} k_j}{m^2} \ll 1 \quad \text{with} \quad k_i \Theta k_j \sim \mathcal{O}(1), \quad (1.2)$$

for external momenta k_i , where m is the mass of the W-bosons stretched between the branes.

Before we begin, let us recall the commutative result. For convenience, we restrict ourselves to the case of two single branes. To lowest order in external momenta, the one-loop amplitude for four gauge bosons can be written in the form (see *e.g.* [23, 24])

$$\Gamma_{\text{1-loop}} \propto \frac{1}{r^4} \int d^4y [F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\kappa F^\kappa{}_\mu - \frac{1}{4} F^\mu{}_\nu F^\nu{}_\mu F^\rho{}_\kappa F^\kappa{}_\rho]. \quad (1.3)$$

where $F = F_1 - F_2$ and the subscripts 1, 2 refer to the two D3-branes.

The factor of $\frac{1}{r^4}$ in equations (1.3) arises in the SYM box diagram via the W-boson mass [$m = \frac{r}{2\pi\alpha'}$] in the propagators. Alternatively, one can understand it as arising via the propagator for exchange of massless closed strings in the transverse space between the branes.

When we turn on a nonvanishing Θ , we find some new interesting phenomena for the nonplanar amplitudes:

- Due to the extra propagating distance $|\Delta x| = |\Theta k|$ for the W-bosons running in the loop, where k now is the momentum flow between the branes, the factor $\frac{1}{r^4}$ is replaced by

$$\frac{1}{m^4} K_2(m |\Delta x|), \quad m = \frac{r}{2\pi\alpha'}. \quad (1.4)$$

where K_2 is a modified Bessel function. In particular, when $m |\Delta x| \gg 1$, the amplitude is exponentially suppressed, proportional to

$$\exp[-m |\Delta x|] \quad (1.5)$$

³The open string metric $G_{\mu\nu}$ and noncommutativity parameter $\Theta^{\mu\nu}$ are related to the closed string metric $g_{\mu\nu}$ and the B field by $G^{\mu\nu} = (\frac{1}{g+B} g \frac{1}{g-B})^{\mu\nu}$ and $\Theta^{\mu\nu} = -2\pi\alpha' (\frac{1}{g+B} B \frac{1}{g-B})^{\mu\nu}$.

typical of an intermediate particle of mass m propagating over a distance $|\Delta x|$. From the closed string point of view, equation (1.4) implies that the closed strings propagating between the branes are massive. Due to different scalings between the closed and open string metrics, the low energy limit on the brane in terms of open string metric $G_{\mu\nu}$ no longer implies the low energy limit for the bulk closed string modes. Thus, in the kinematic regime (1.2), there is a significant longitudinal momentum transfer between the branes, and the closed string modes indeed obtain an effective mass $M = \frac{|\Delta x|}{2\pi\alpha'} = \frac{|\Theta k|}{2\pi\alpha'}$.

- We have attempted to write down a one-loop effective action, from integrating out massive W -bosons, in terms of a derivative expansion in the regime (1.2). We find that while it is possible to formally write down an off-shell effective action from the gauge invariant on-shell amplitudes, the off-shell action is not gauge invariant in terms of the noncommutative $F_{\mu\nu}$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu * A_\nu + iA_\nu * A_\mu, \quad (1.6)$$

defined at tree-level. The technical reason is the following. In nonplanar processes, writing an effective action requires additional $*$ -operations which are not gauge invariant under the gauge transformations of the noncommutative F .⁴ For example, for the $2 \rightarrow 2$ process of figure 1, the product between two photon modes on the same worldsheet boundary becomes

$$F^{\mu\nu} *' F_{\mu\nu}, \quad (1.7)$$

where the $*'$ -product is given by

$$\begin{aligned} f(x) *' g(x) &\equiv f(x) \frac{\sin\left(\frac{1}{2}\Theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right)}{\frac{1}{2}\Theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu} g(x), \\ &= f(x)g(x) - \frac{1}{24}\Theta^{\mu\nu}\Theta^{\sigma\tau}\partial_\mu\partial_\sigma f(x)\partial_\nu\partial_\tau g(x) + \dots. \end{aligned} \quad (1.8)$$

The result is somewhat surprising. From the field theory point of view, we have integrated out the massive W -bosons. According to conventional wisdom, we should be able to write down a gauge-invariant effective action as a derivative expansion. That it is no longer possible to do so might be another indication of IR/UV mixing in noncommutative theories, even though in this case we do not have the IR singularities.

⁴For the full expression for the effective action, see equation (3.23).

The paper is organized as follows. In section 2, we evaluate the string amplitude. The field theory limit is explored in section 3. In section 4, we conclude with a discussion of gauge invariance. We describe our treatment of the fermionic string coordinates in the presence of a B -field, used in section 2, in appendix A.

2. Correlation Functions

We are interested in computing the one-loop scattering amplitudes of “brane waves” on parallel Dp -branes in type II string theory—specifically, the four gauge boson annulus amplitude, both planar and nonplanar (*e.g.* figure 1).

On the annulus, we can work in the “zero” picture, for which the relevant vertex operator is [25]

$$\mathcal{V}^0 = \frac{g_o}{\sqrt{2\alpha'}} T^a e_\mu^a (i\dot{X}^\mu + 4k \cdot \Psi \Psi^\mu) e^{ik \cdot X}. \quad (2.1)$$

In equation (2.1), T^a is a generator of the gauge group, e_μ^a is the polarization of the gluon and Ψ^μ is the linear combination of the left and right-moving fermions, ψ and $\tilde{\psi}$, that lives on the boundary (see appendix A). The dot product is simple contraction: $k \cdot \Psi \equiv k_\mu \Psi^\mu$. For general periodicity $\psi^\mu(w + 2n\pi + 2m\pi it) = e^{2\pi i[n(\alpha-1/2)+m(\beta-1/2)]} \psi^\mu(w)$, the Green functions for Ψ can be written

$$\langle \Psi^\mu(w_1) \Psi^\nu(w_2) \rangle = \frac{\alpha'}{4\pi} G^{\mu\nu} \frac{\vartheta[\frac{\alpha}{\beta}](\frac{w_1-w_2}{2\pi}|it) \vartheta'[\frac{1/2}{1/2}](0|it)}{\vartheta[\frac{1/2}{1/2}](\frac{w_1-w_2}{2\pi}|it) \vartheta[\frac{\alpha}{\beta}](0|it)}, \quad (A.9'')$$

and the partition function is

$$\langle 1 \rangle = \frac{\vartheta[\frac{\alpha}{\beta}](0|it)^4}{\eta(it)^4}. \quad (A.8')$$

It is well known (*see e.g.* [25]) that, on the annulus, the first nonvanishing correlation function involves four pairs of fermions. This gives the one-loop nonrenormalization theorem for the F^2 and F^3 terms in the effective supersymmetric spacetime action. Because the effect of the B -field on the fermions can be absorbed into the doubling trick, this continues to hold.⁵ Thus, the first nontrivial amplitude on the annulus involves four gauge bosons, and the corresponding amplitude is

$$\begin{aligned} & \sum_{\pm, \pm'} \frac{g_o^4}{4\alpha'^2} (2\alpha')^4 T^a T^b T^c T^d e_{1\mu}^a e_{2\nu}^b e_{3\rho}^c e_{4\sigma}^d \langle e^{ik_1 \cdot X}(w_1) e^{ik_2 \cdot X}(w_2) e^{ik_3 \cdot X}(w_3) e^{ik_4 \cdot X}(w_4) \rangle \\ & \times \langle :k_1 \cdot \Psi \Psi^\mu:(w_1) :k_2 \cdot \Psi \Psi^\nu:(w_2) :k_3 \cdot \Psi \Psi^\rho:(w_3) :k_4 \cdot \Psi \Psi^\sigma:(w_4) \rangle_{\pm\pm'} \end{aligned} \quad (2.2)$$

⁵It is easy to check this, using equations (A.8') and (A.9''), and the Jacobi fundamental formula for sums of products of ϑ -functions [26, p. 467–468] of which the abstruse identity is a special case.

where we have factored the amplitude into fermionic and bosonic factors, and suppressed the ghosts. The sum is over spin structures. The fermionic contribution is easily found to be

$$\begin{aligned} \sum_{\pm, \pm'} \langle :k_1 \cdot \Psi \Psi^\mu : (w_1) :k_2 \cdot \Psi \Psi^\nu : (w_2) :k_3 \cdot \Psi \Psi^\rho : (w_3) :k_4 \cdot \Psi \Psi^\sigma : (w_4) \rangle_{\pm \pm'} \\ = \eta(it)^8 k_{1\tau} k_{2\lambda} k_{3\alpha} k_{4\beta} t^{\tau\mu\lambda\nu\alpha\rho\beta\sigma}, \end{aligned} \quad (2.3)$$

where $t^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4}$ is antisymmetric in $\alpha_i \leftrightarrow \beta_i$ (for each i) and is symmetric in $(\alpha_i, \beta_i) \leftrightarrow (\alpha_j, \beta_j)$. It is convenient to define

$$\mathcal{K} \equiv k_{1\alpha_1} e_{1\beta_1}^a k_{2\alpha_2} e_{2\beta_2}^b k_{3\alpha_3} e_{3\beta_3}^c k_{4\alpha_4} e_{4\beta_4}^d t^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\alpha_4\beta_4} \quad (2.4a)$$

which gives

$$\mathcal{K} = M_{1\mu}{}^\lambda M_{2\lambda}{}^\alpha M_{3\alpha}{}^\nu M_{4\nu}{}^\mu - \frac{1}{4} M_{1\mu\nu} M_2{}^{\nu\mu} M_{3\tau\sigma} M_4{}^{\sigma\tau} + 2 \text{ permutations.} \quad (2.4b)$$

In (2.4), $M_{i\mu\nu} = e_{i\mu} k_{i\nu} - k_{i\mu} e_{i\nu}$, the indices are raised and lowered using the open string metric G , and the two permutations are given by changing the ordering from (1234) to (1342) and (1423) (see *e.g.* [25]).

The bosonic contribution is identical to the tachyon correlation function computed for the bosonic string in [27, 28, 29, 30, 18, 31]. Combining equations (2.3) and [18, eq. (2.15)] gives the total amplitude

$$\begin{aligned} \mathcal{A} = -i\sqrt{\det G} \frac{g_o^4}{4\alpha'^2} (2\alpha')^4 (2\pi)^{p+1} \delta^{(p+1)} \left(\sum_{q=1}^4 k_q \right) \mathcal{K} \\ \times \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\frac{p+1}{2}} \exp \left[\frac{k_\mu (\Theta G \Theta)^{\mu\nu} k_\nu}{8\pi \alpha' t} - \frac{r^2 t}{2\pi \alpha'} \right] \left[\prod_{q=1}^4 \int_0^{2\pi t} d\tau_q \right] \Psi_1 \Psi_2 \Psi_{12} \\ \times \exp \left[-\frac{i}{2} \sum_{i < j} (k_i \times k_j) \left[\frac{\tau_{ij}}{\pi t} - \epsilon(\tau_{ij}) \right] + \frac{i}{2} \sum_{y < z} (k_y \times k_z) \left[\frac{\tau_{yz}}{\pi t} - \epsilon(\tau_{yz}) \right] \right] \end{aligned} \quad (2.5)$$

where $\tau_{ij} = \tau_i - \tau_j$, $k = \sum_i k_i = -\sum_y k_y$ is the nonplanar momentum, $\epsilon(\tau)$ is the sign of τ , and

$$\Psi_1 = \prod_{i < j} |\psi_{ij}|^{2\alpha' k_{i\mu} G^{\mu\nu} k_{j\nu}}, \quad \Psi_2 = \prod_{y < z} |\psi_{yz}|^{2\alpha' k_{y\mu} G^{\mu\nu} k_{z\nu}}, \quad \Psi_{12} = \prod_{i,y} (\psi_{iy}^T)^{2\alpha' k_{y\mu} G^{\mu\nu} k_{i\nu}}, \quad (2.6a)$$

$$\psi_{ij} = 2\pi i \exp \left(-\frac{\tau_{ij}^2}{4\pi t} \right) \frac{\vartheta_1(i\frac{\tau_{ij}}{2\pi}|it)}{\vartheta'_1(0|it)}, \quad \psi_{iy}^T = 2\pi \exp \left(-\frac{\tau_{iy}^2}{4\pi t} \right) \frac{\vartheta_2(i\frac{\tau_{iy}}{2\pi}|it)}{\vartheta'_1(0|it)}. \quad (2.6b)$$

Here, i and y are indices which run over the vertex operators on the two boundaries, and q runs over all vertex operators. The quantity \mathcal{K} was given in (2.4), and we have suppressed Chan-Paton factors. We also have introduced a transverse distance r between the parallel branes. It gives a mass

$$m = \frac{r}{2\pi\alpha'} \quad (2.7)$$

to the ground states of open strings stretching between the branes and acts as an infrared cutoff in the world-volume Yang-Mills theory.

The exponential factor

$$\exp \left[\frac{1}{8\pi\alpha' t} k_\mu (\Theta G \Theta)^{\mu\nu} k_\nu \right] = \exp \left[-\frac{(\Delta x)^2}{8\pi\alpha' t} \right], \quad \Delta x^\mu \equiv \Theta^{\mu\nu} k_\nu, \quad (2.8)$$

in (2.5), which was interpreted in [18] as the manifestation of a stretched string in the bosonic string theory, and was responsible for IR/UV mixing in those theories, survives the field theory limit in the superstring as well. Thus, the observations made in [18] apply also here.

3. One-loop Amplitude in Noncommutative $\mathcal{N} = 4$ SYM

The one-loop amplitude for noncommutative $\mathcal{N} = 4$ super-Yang-Mills theory can be obtained from (2.5) by taking $\alpha' \rightarrow 0$ with $\alpha' t$ fixed. While our main focus is on the 3+1-dimensional $\mathcal{N} = 4$ noncommutative SYM theory, our results apply to generic p , and we will keep our discussion general. To take the field theory limit, we have been implicitly considering only magnetic B ; however, the results of section 2 are completely general, holding also for electric B , and so can be used in the context of the NCOS [4, 5, 6, 7, 8]. Note that as $t \rightarrow \infty$,

$$\psi_{ij} = -2 \sinh \left(\frac{\tau_{ij}}{2} \right) \exp \left(-\frac{\tau_{ij}^2}{4\pi t} \right), \quad \psi_{iy}^T = 2 \cosh \left(\frac{\tau_{iy}}{2} \right) \exp \left(-\frac{\tau_{iy}^2}{4\pi t} \right). \quad (3.1)$$

To isolate the t dependence we rescale τ so that $0 < \tau < 1$. Let $T = 2\pi m^2 \alpha' t$ and recall that $g_{\text{YM}} = g_o / \sqrt{2\alpha'}$. This gives the field theory amplitude

$$\begin{aligned} \mathcal{A} = & -i 2^4 \pi^{\frac{p+1}{2}} \sqrt{\det G} g_{\text{YM}}^4 \delta^{(p+1)} \left(\sum_{q=1}^4 k_q \right) \mathcal{K} m^{p-7} \int_0^\infty \frac{dT}{2T} T^{\frac{7-p}{2}} \exp \left[-\frac{m^2 (\Delta x)^2}{4T} - T \right] \\ & \times \left[\prod_{q=1}^4 \int_0^1 d\tau_q \right] \exp \left[-\frac{i}{2} \sum_{i < j} (k_i \times k_j) [2\tau_{ij} - \epsilon(\tau_{ij})] + \frac{i}{2} \sum_{y < z} (k_y \times k_z) [2\tau_{yz} - \epsilon(\tau_{yz})] \right] \\ & \times \exp \left[T \sum_{p < q} \frac{k_{p\mu} G^{\mu\nu} k_{q\nu}}{m^2} [|\tau_{pq}| - \tau_{pq}^2] \right]. \quad (3.2) \end{aligned}$$

Later we shall examine the amplitude (3.2) in the kinematic region $k_{p\mu}G^{\mu\nu}k_{q\nu} \ll m^2$, with $k_p\Theta k_q$ remaining finite [equation (1.2)].⁶ In this case the last factor of (3.2) has the expansion

$$\left[1 + T \sum_{p < q} \frac{k_{p\mu}G^{\mu\nu}k_{q\nu}}{m^2} [|\tau_{pq}| - \tau_{pq}^2] + \dots \right]. \quad (3.3)$$

Let us first look at the planar contribution. In this case $k = 0$ and $\Delta x = 0$. Also, $\sum_{i < j} (k_i \times k_j) \tau_{ij} = 0$ by momentum conservation; thus the phase factor in (3.2) is just

$$\exp \left[\frac{i}{2} \sum_{p < q} (k_p \times k_q) \epsilon(\tau_{pq}) \right]. \quad (3.4)$$

This turns the amplitude

$$\mathcal{A} \sim \frac{1}{m^4} \int d^4x \text{Tr} F^4 + \frac{1}{m^2} \int d^4x \text{Tr} F F D F D F + \dots \quad (3.5)$$

of the commutative theory into the

$$\mathcal{A} \sim \frac{1}{m^4} \int d^4x \text{Tr}(F * F * F * F) + \frac{1}{m^2} \int d^4x \text{Tr}(F * F * D F * D F) + \dots \quad (3.6)$$

of the noncommutative theory. (Here we have been schematic and have not included the Lorentz indices.)

The story for nonplanar diagrams is more complicated. There are essentially two new features for nonplanar diagrams in (3.2) compared to $B = 0$ and the planar diagram. The first is the appearance, in the integral over proper time T , of $(\Delta x)^2$, in a manner which suppresses the contribution to the amplitude from small T (short-distance). The second new feature is that in the integration of the vertex operator coordinates, there are additional phase factors

$$\exp \left[-i \sum_{i < j} (k_i \times k_j) (\tau_i - \tau_j) \right]. \quad (3.7)$$

Both features already appeared in the bosonic cases [27, 29, 18]. Here with a large number of supersymmetries present, their effects can be isolated and subjected to a thorough analysis.

⁶We can also define $T = \frac{2\pi\alpha' t}{(\Delta x)^2}$ and consider an expansion in terms of $k^2(\Delta x)^2$. Either expansion gives the same leading order result, with which we are mostly concerned.

3.1 Stretched strings and closed strings

The factor $\exp\left[-\frac{m^2(\Delta x)^2}{4T}\right]$, in equation (3.2), was interpreted in [18] as the effect of a stretched string. More explicitly, when $B \neq 0$, an open string has a nonzero length, as described in the introduction, the effect of which is to cut open an otherwise closed first-quantized particle loop. The exponential factor may be understood as the amplitude for diffusion of a mass m particle over a distance $|\Delta x|$. When, for $B = 0$, the loop integral is divergent at short distances (small T), the stretched string effect regularizes the divergence. However, taking Δx to zero will recover the original UV divergence. This is the origin of the IR/UV mixing discussed in [11]. In theories for which the commutative loop integral is convergent in the UV region, we would expect that the $\Delta x \rightarrow 0$ limit can be achieved in a continuous way, and that IR singularities are absent. It might be expected that the effect of the stretched string is mild in these cases. However, as we shall see below, it suppresses the amplitude exponentially at large external k .

In the low energy expansion in terms of $\frac{k_p k_q}{m^2}$ the integration over T can be performed exactly for each term in the series and gives [32, eq. 8.432.6]

$$I_n = m^{p-7} \int_0^\infty \frac{dT}{2T} T^{\frac{7-p}{2}+n} e^{-\frac{m^2(\Delta x)^2}{4T}-T} = \left(\frac{m|\Delta x|}{2}\right)^n \left(\frac{2m}{|\Delta x|}\right)^{\frac{p-7}{2}} K_{\frac{7-p}{2}+n}(m|\Delta x|), \quad (3.8)$$

where K is the modified Bessel function. Recall that $K_{-\nu}(z) = K_\nu(z)$ and as $z \rightarrow 0$,

$$K_\nu(z) = \begin{cases} \frac{\Gamma(\nu)}{2} \left(\frac{2}{z}\right)^\nu, & \nu > 0, \\ -\ln \frac{z}{2}, & \nu = 0. \end{cases} \quad (3.9)$$

For generic p and n , as $\Delta x \rightarrow 0$, we have

$$I_n \sim \begin{cases} \frac{\Gamma(n+\frac{7-p}{2})}{2} m^{p-7}, & n + \frac{7-p}{2} > 0, \\ -\log m |\Delta x|, & p = 7, n = 0, \\ \frac{\Gamma(\frac{7-p}{2})}{2} \left(\frac{2}{|\Delta x|}\right)^{p-7}, & p = 8, 9; n = 0, \\ -m^2 \log(m|\Delta x|), & p = 9, n = 1. \end{cases} \quad (3.10)$$

When $n + \frac{7-p}{2} > 0$, the integral in (3.8) is convergent as $\Delta x = 0$; thus, as $\Delta x \rightarrow 0$, I_n reduces to the standard results for $B = 0$. In the other cases, logarithmic, linear and quadratic divergences, respectively for 8, 9 and 10-dimensional SYM theory, are reflected in the singular dependence on Δx in the amplitude.

For $m |\Delta x| \gg 1$ —*i.e.* the large external momentum limit—

$$I_n \sim \exp[-m |\Delta x|]. \quad (3.11)$$

The amplitude is exponentially suppressed at large k . The behaviour in (3.11) is easily understood from the stretched string picture in which a particle of mass m diffuses a distance $|\Delta x|$. This large momentum suppression is reminiscent of the asymptotic behaviour of correlation functions observed in the supergravity side [33] and the high temperature suppression of the partition function for nonplanar diagrams of $\mathcal{N} = 4$ noncommutative SYM [34], although the details are somewhat different.

We note that for $n = 0$ —*i.e.* the leading term in the expansion (3.3)—there is also an interpretation for (3.8) in terms of the lowest closed string modes. Defining $M^2 = (\frac{\Delta x}{2\pi\alpha'})^2$ and $\tilde{d} = 9 - p$, the codimension of the brane, we can rewrite equation (3.8) as (notice that $Mr = m |\Delta x|$)

$$I_0 = (2\pi\alpha')^{7-p} 2\pi^{\frac{9-p}{2}} \int \frac{d^{\tilde{d}}q}{(2\pi)^{\tilde{d}}} \frac{e^{iq \cdot r}}{q^2 + M^2}. \quad (3.12)$$

The right hand side of equation (3.12) has the interpretation of a massive particle of mass M propagating between the branes. In contrast with the $B = 0$ case ($\Delta x \rightarrow 0$ in (3.8)), for which the propagator is $m^{p-7} = (\frac{2\pi\alpha'}{r})^{7-p}$ corresponding to a virtual mediating massless closed string, now the propagating particle has a mass $M = \frac{|\Delta x|}{2\pi\alpha'}$. This result can be understood by noting that the intermediate closed string has momentum k_μ (defined below equation (2.5)) along the brane direction from momentum conservation and therefore an effective mass

$$M^2 = -k_\mu g^{\mu\nu} k_\nu = -k_\mu G^{\mu\nu} k_\nu + \left(\frac{1}{2\pi\alpha'}\right)^2 k_\mu (\Theta G \Theta)^{\mu\nu} k_\nu \sim \left(\frac{\Delta x}{2\pi\alpha'}\right)^2, \quad (3.13)$$

where in the last step we have recalled equation (1.2). Physically, this means that due to the different scaling between the closed and open string metrics, the low energy limit for the brane world-volume theory does not correspond to the low energy limit for the bulk closed string modes.

It is interesting to compare the IR/UV relation between the open and closed string pictures. For $p \leq 6$, SYM theory is IR divergent when $m = 0$; this is reflected in the UV divergence, for codimension $\tilde{d} \geq 3$, of the transverse propagator on the closed string side, as we reduce the separation of the branes to zero. For $p = 8, 9$, the SYM theory is linearly and quadratically UV divergent while the transverse propagator is IR divergent ($M \rightarrow 0$) in codimension $\tilde{d} = 1, 0$. At $p = 7$ SYM theory is both UV and IR divergent and the closed string propagator in codimension 2 is both IR and UV divergent.

The UV/IR relations in equation (3.10) for $p = 7, 8, 9$ are precise realizations of closed string idea proposed in [11, 12] to explain IR/UV mixing in noncommutative theories. However we note that that picture seems to apply only for the $n = 0$ term, which is protected by a nonrenormalization theorem from contributions of massive string modes [35]. For higher order terms, in principle, there can be an infinite number of massive closed string modes involved. For example, the $p = 9, n = 1$ term is logarithmically divergent as $\Delta x \rightarrow 0$, but there is no propagation of a codimension two closed string to explain it.⁷

3.2 Beyond the $*$ -product

Now we would like to focus on the leading order ($n = 0$) terms, and attempt to write down an off-shell gauge invariant one-loop effective action.

Due to the presence of the additional phase factors,

$$\exp \left[-i \sum_{i < j} (k_i \times k_j) (\tau_i - \tau_j) \right], \quad (3.14)$$

in the integrals over τ_q in (3.2), the product between the insertions on the same boundary is neither an ordinary nor a $*$ -product, as it was for the $B = 0$, and planar cases, respectively. In particular and interestingly, the nonplanar terms in the action are not $*$ -product generalizations of the familiar ones.

For the nonplanar diagram with two vertex operators on each boundary (we label 1, 2 on one boundary and 3, 4 on the other), we find that the τ -integrations give rise (including the Chan-Paton factors) to a factor of

$$\text{Tr}(T^a T^b) \text{Tr}(T^c T^d) \frac{\sin\left(\frac{k_1 \times k_2}{2}\right)}{\frac{k_1 \times k_2}{2}} \frac{\sin\left(\frac{k_3 \times k_4}{2}\right)}{\frac{k_3 \times k_4}{2}}. \quad (3.15)$$

Note that (3.15) is finite as $k_1 \times k_2 \rightarrow 0$ and/or $k_3 \times k_4 \rightarrow 0$.

The other type of nonplanar diagram has 1, 2, 3 on one boundary and 4 on the other. Then, for the ordering $\tau_1 > \tau_2 > \tau_3$,

$$\begin{aligned} & \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{\frac{i}{2}(k_1 \times k_2 + k_1 \times k_3 + k_2 \times k_3)} e^{-i(k_1 \times k_2)\tau_{12} - i(k_1 \times k_3)\tau_{13} - i(k_2 \times k_3)\tau_{23}} \\ &= \frac{e^{\frac{i}{2}(k_1 \times k_2 + k_3 \times k_4)}}{(k_1 \times k_4)(k_3 \times k_4)} + 2 \frac{e^{\frac{i}{2}k_2 \times k_3} \sin\left(\frac{k_1 \times k_4}{2}\right)}{(k_1 \times k_4)^2 (k_2 \times k_4)} + 2 \frac{e^{\frac{i}{2}k_1 \times k_2} \sin\left(\frac{k_3 \times k_4}{2}\right)}{(k_2 \times k_4)^2 (k_3 \times k_4)}. \end{aligned} \quad (3.16)$$

⁷The closed string proposal also suffers problems at two loops.[22]

This is nonsingular even when $k_i \times k_j \rightarrow 0$. Including all the orderings, as well as the minus sign that comes with the odd-number of Chan-Paton matrices on the $\sigma = \pi$ boundary, gives

$$\begin{aligned}
& -\text{Tr}(T^b T^a T^c) \text{Tr}(T^d) \left\{ \frac{e^{\frac{i}{2}(k_1 \times k_2 + k_1 \times k_3 + k_2 \times k_3)}}{[k_3 \times (k_1 + k_2)][k_1 \times (k_2 + k_3)]} + \frac{e^{\frac{i}{2}(k_2 \times k_3 - k_1 \times k_2 - k_1 \times k_3)}}{[k_1 \times (k_2 + k_3)][k_2 \times (k_1 + k_3)]} \right. \\
& \quad \left. + \frac{e^{\frac{i}{2}(k_1 \times k_2 - k_1 \times k_3 - k_2 \times k_3)}}{[k_2 \times (k_1 + k_3)][k_3 \times (k_1 + k_2)]} \right\} \\
& -\text{Tr}(T^a T^b T^c) \text{Tr}(T^d) \left\{ \frac{e^{\frac{i}{2}(k_2 \times k_3 + k_1 \times k_3 - k_1 \times k_2)}}{[k_3 \times (k_1 + k_2)][k_2 \times (k_1 + k_3)]} + \frac{e^{\frac{i}{2}(k_1 \times k_3 + k_1 \times k_2 - k_2 \times k_3)}}{[k_2 \times (k_1 + k_3)][k_1 \times (k_2 + k_3)]} \right. \\
& \quad \left. + \frac{e^{-\frac{i}{2}(k_1 \times k_2 + k_1 \times k_3 + k_2 \times k_3)}}{[k_1 \times (k_2 + k_3)][k_3 \times (k_1 + k_2)]} \right\}. \quad (3.17)
\end{aligned}$$

When the gauge group is abelian, equation (3.17) can be simplified to⁸

$$-\frac{\sin\left(\frac{k_2 \times k_3}{2}\right) \sin\left(\frac{k_1 \times (k_2 + k_3)}{2}\right)}{\frac{(k_1 + k_2) \times k_3}{2} \frac{k_1 \times (k_2 + k_3)}{2}} - \frac{\sin\left(\frac{k_1 \times k_3}{2}\right) \sin\left(\frac{k_2 \times (k_1 + k_3)}{2}\right)}{\frac{(k_1 + k_2) \times k_3}{2} \frac{k_2 \times (k_1 + k_3)}{2}}. \quad (3.18)$$

If we have N vertex operators on the same boundary, the general product structure between the operators is given by

$$\text{Tr} [T^{a_1} \dots T^{a_N}] \left[\prod_{q=1}^N \int_0^{\tau_{q-1}} d\tau_q \right] \exp \left[-\frac{i}{2} \sum_{i < j} (k_i \times k_j) [2\tau_{ij} - \epsilon(\tau_{ij})] \right]; \quad \tau_0 \equiv 1, \quad (3.19)$$

plus permutations.

To lowest order in k^2/m^2 , the four gluon scattering amplitudes are given by

$$\begin{aligned}
\mathcal{A}_{2 \rightarrow 2} = & -i 2^4 \pi^{\frac{p+1}{2}} \sqrt{\det G} g_{\text{YM}}^4 \delta^{(p+1)} \left(\sum_{q=1}^4 k_q \right) \text{Tr}(T^a T^b) \text{Tr}(T^c T^d) \mathcal{K} \\
& \times \left(\frac{\Delta x}{2m} \right)^{\frac{7-p}{2}} K_{\frac{7-p}{2}}(m \Delta x) \frac{\sin\left(\frac{k_1 \times k_2}{2}\right)}{\frac{k_1 \times k_2}{2}} \frac{\sin\left(\frac{k_3 \times k_4}{2}\right)}{\frac{k_3 \times k_4}{2}}, \quad (3.20)
\end{aligned}$$

where \mathcal{K} is given by (2.4). Similarly (for simplicity, we only give the $U(1)$ result in this

⁸This is not manifestly symmetric in the three vertex operators on one boundary, but follows from the manifestly symmetric expression (3.17).

case),

$$\begin{aligned} \mathcal{A}_{1 \rightarrow 3} &= i 2^4 \pi^{\frac{p+1}{2}} \sqrt{\det G} g_{\text{YM}}^4 \delta^{(p+1)} \left(\sum_{q=1}^4 k_q \right) \mathcal{K} \\ &\times \left(\frac{\Delta x}{2m} \right)^{\frac{7-p}{2}} K_{\frac{7-p}{2}}(m \Delta x) \left[\frac{\sin\left(\frac{k_2 \times k_3}{2}\right) \sin\left(\frac{k_1 \times (k_2 + k_3)}{2}\right)}{\frac{(k_1 + k_2) \times k_3}{2} \frac{k_1 \times (k_2 + k_3)}{2}} + \frac{\sin\left(\frac{k_1 \times k_3}{2}\right) \sin\left(\frac{k_2 \times (k_1 + k_3)}{2}\right)}{\frac{(k_1 + k_2) \times k_3}{2} \frac{k_2 \times (k_1 + k_3)}{2}} \right]. \end{aligned} \quad (3.21)$$

It is easy to see that the above amplitudes are gauge invariant on-shell; *i.e.* $\mathcal{A} = 0$, when we take $e_i \propto k_i$.

Now we would like to extend the above on-shell amplitudes to the off-shell effective action. The standard way of doing this is to take $M_{\mu\nu} = e_\mu k_\nu - e_\nu k_\mu \rightarrow -\frac{i}{g_{\text{YM}}} F_{\mu\nu}$ with $F_{\mu\nu}$ given by the tree-level expression

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i A_\mu * A_\nu + i A_\nu * A_\mu \quad (3.22)$$

Combining these expressions, we find that the 1-loop effective action contains the terms⁹

$$\begin{aligned} \Gamma_{\text{1-loop}} &= -\frac{1}{4!(4\pi)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{\det G} t^{\mu\nu\rho\sigma\lambda\tau\alpha\beta} \\ &\times \left\{ \frac{N_2}{2} \Gamma\left(\frac{7-p}{2}\right) \frac{1}{m^{7-p}} \text{Tr}_{U(N_1)} [F_{\mu\nu}(x) * F_{\rho\sigma}(x) * F_{\lambda\tau}(x) * F_{\alpha\beta}(x)] \right. \\ &+ 3 \text{Tr}_{U(N_1)} [F_{\mu\nu}(x) *' F_{\rho\sigma}(x)] I_{\frac{7-p}{2}} \left(m, \sqrt{\overleftarrow{\partial}_\mu (\Theta G \Theta)^{\mu\nu} \overrightarrow{\partial}_\nu} \right) \text{Tr}_{U(N_2)} [F_{\lambda\tau}(x) *' F_{\alpha\beta}(x)] \\ &- 4 \text{Tr}_{U(N_1)} (*_3 [F_{\mu\nu}(x), F_{\rho\sigma}(x), F_{\lambda\tau}(x)]) I_{\frac{7-p}{2}} \left(m, \sqrt{-\partial_\mu (\Theta G \Theta)^{\mu\nu} \partial_\nu} \right) \text{Tr}_{U(N_2)} (F_{\alpha\beta}(x)) \Big\} \\ &+ (N_1 \rightarrow N_2) \end{aligned} \quad (3.23a)$$

where $N = N_1 + N_2$ is the total number of D-branes with N_1 and N_2 D-branes on each boundary; the factor of $N_{1,2}$ appears in the planar diagram from tracing over the

⁹Using equation (2.4), we can write *e.g.* (up to a total derivative if $*$ -products are used)

$$\begin{aligned} &t^{\mu\nu\rho\sigma\lambda\tau\alpha\beta} \text{Tr} (F_{\mu\nu}(x) F_{\rho\sigma}(x) F_{\lambda\tau}(x) F_{\alpha\beta}(x)) \\ &= 2^4 \text{Tr} \left(2 F_{\mu\rho} F_\nu^\rho F^{\mu\sigma} F^\nu_\sigma + F_\mu^\rho F_\rho^\nu F_\nu^\sigma F_\sigma^\mu - \frac{1}{2} F_{\mu\rho} F^{\mu\rho} F_{\nu\sigma} F^{\nu\sigma} - \frac{1}{4} F_{\mu\rho} F_{\nu\sigma} F^{\mu\rho} F^{\nu\sigma} \right) \end{aligned}$$

Chan-Paton factors on the empty boundary; the traces are taken in the fundamental of the indicated subgroup;

$$I_n(m, x) \equiv \left(\frac{x}{2m} \right)^n K_n(mx); \quad (3.23b)$$

and we have defined, from (3.15), the $*'$ -product (1.8), and from (3.17), the $*_3$ -ternary operation. For an abelian gauge group, the $*'_3$ -ternary operation

$$\begin{aligned} *_3' [f(x), g(x), h(x)] \\ \equiv \left[\frac{\sin\left(\frac{\partial_2 \times \partial_3}{2}\right) \sin\left(\frac{\partial_1 \times (\partial_2 + \partial_3)}{2}\right)}{\frac{(\partial_1 + \partial_2) \times \partial_3}{2} \frac{\partial_1 \times (\partial_2 + \partial_3)}{2}} + \frac{\sin\left(\frac{\partial_1 \times \partial_3}{2}\right) \sin\left(\frac{\partial_2 \times (\partial_1 + \partial_3)}{2}\right)}{\frac{(\partial_1 + \partial_2) \times \partial_3}{2} \frac{\partial_2 \times (\partial_1 + \partial_3)}{2}} \right] f(x_1)g(x_2)h(x_3)|_{x_i=x}, \end{aligned} \quad (3.24)$$

defined following (3.18), gives the same result as the similarly defined $*_3$ -ternary operation. For the general product (3.19) we can define a $*_n$ n -ary operation which we will not write down explicitly.

We should note that (for $p < 7$) equations (3.8) and (3.10) imply that the power expansion of $I_{\frac{7-p}{2}}(m, x)$ contains only nonnegative, even integer powers of x ; thus equation (3.23a) contains only nonnegative integer powers of derivatives, and so is well-defined. The limit $\Theta \rightarrow 0$ is also well-defined—for example, all the $*_n$ n -ary operations become ordinary products—and recovers the nonabelian analogue of (1.3) given in *e.g.* [36] and references therein.

We have some remarks regarding equation (3.23):

- The $*'$ -product was also found by Garousi in [37]. When considering the scattering amplitudes of one closed string with two open string modes,¹⁰ he found that the $*'$ -product should be used to multiply the massless open string fields.
- The nonplanar part of equation (3.23) is not gauge-invariant, off-shell. To be specific, we focus on the second line of (3.23a); the symmetry of the expression—and particularly that of the $*'$ -product—implies that this term transforms into

$$\begin{aligned} 6i \text{Tr}_{U(N_1)}[(\Lambda * F_{\mu\nu} - F_{\mu\nu} * \Lambda) *' F_{\rho\sigma}] I_{\frac{7-p}{2}} \left(m, \sqrt{\overleftarrow{\partial}_\mu (\Theta G \Theta)^{\mu\nu} \overrightarrow{\partial}_\nu} \right) \text{Tr}_{U(N_2)}[F_{\lambda\tau} *' F_{\alpha\beta}] \\ + (N_1 \leftrightarrow N_2). \end{aligned} \quad (3.25)$$

At $B = \Theta = 0$, the $*$ - and $*'$ -products reduce to the ordinary product, and so the expression (3.25) vanishes by cyclicity of the trace and the symmetry

¹⁰These disk diagrams were also considered in [38].

properties of the tensor t . For the planar contribution [the first term of (3.23a)] the associativity of the $*$ -product, combined with cyclicity of the trace, causes that term to be gauge invariant. Generically the total 1-loop effective action is not gauge invariant.

- In the commutative limit, $\text{Tr } F^2$ and $\text{Tr } F^3$ (with appropriate Lorentz contractions) are gauge invariant operators, in (3.23a), which couple to closed string modes in the bulk. They are also observables used in the AdS/CFT correspondence. However in the noncommutative theory, the corresponding operators in (3.23) are no longer gauge invariant. This is hardly surprising, since there are no gauge invariant local operators in the theory [14, 15, 39]. In [15, 39], gauge invariant observables in NCSYM corresponding to supergravity modes [40, 33] were constructed using open Wilson lines [41, 42, 43].¹¹ It is natural to wonder whether it is possible to replace (3.23) by a gauge invariant version in terms of these observables or their generalizations. Our efforts in this direction have not yielded a positive answer.
- $*'_3$ generalizes the $*$ -product above. In fact, the products in equations (3.17) and (3.18) do not appear to separate into pairs—this may be related to the fact that the $*$ -product is not associative.¹² Thus, it seems more natural to talk about a generalized $*_n$ n -ary *operation* rather than a $*$ -product. For example, $*'_3$ defined in (3.24) may be called a ternary operation.
- Our above discussions have been restricted to the leading term in (3.2). For sub-leading terms the τ -integration will generate a much more complicated product pattern. For example the second term in (3.3) gives rise to

$$\left[\prod_{q=1}^4 \int_0^1 d\tau_q \right] \exp \left\{ -\frac{i}{2} \sum_{i < j} (k_i \times k_j) [2\tau_{ij} - \epsilon(\tau_{ij})] + \frac{i}{2} \sum_{y < z} (k_y \times k_z) [2\tau_{yz} - \epsilon(\tau_{yz})] \right\} \sum_{p < q} [|\tau_{pq}| - \tau_{pq}^2] \quad (3.26)$$

¹¹Specifically, although Wilson loops are no longer gauge invariant in a noncommutative field theory, the references show that if a Wilson loop of definite momentum is “cut open” so that its endpoints are separated by precisely the displacement of the stretched string, then the resultant (nonlocal) Wilson line is gauge invariant.

¹²The nonassociativity also means that we cannot gauge, in the sense of [44, 45] (see also [46]), the $*$ -product to the ordinary product.

Thus it is a generic phenomenon that, for nonplanar diagrams, the amplitudes are no longer expressible in terms of $*$ -products alone, and may not be extended off-shell in a gauge invariant way using (3.22).

4. Discussion and Conclusions

We have shown that for noncommutative $\mathcal{N} = 4$ SYM at one-loop, just as for the commutative theory, there is no contribution to the F^2 and F^3 terms at one-loop (see also [47]). This simply results from properties of fermion correlation functions on the worldsheet. We also expect that the planar part of the F^4 term is not renormalized beyond one-loop as in the commutative case [48], but this is much less clear for the nonplanar part.

We have written down an off-shell effective action which reproduces the on-shell amplitudes (3.20) and (3.21). This naïve off-shell extension, equation (3.23), is not gauge invariant.¹³ Though equation (3.23) is only the first term in an expansion in $k \cdot G \cdot k$ —while exact in Θ —from the last remark at the end of section 3.2 we expect that the lack of gauge invariance is generic for the higher order terms in the expansion (3.3).

The reason may be attributed to the fact that even in the field theory limit, the theory is stringy, as a result of the stretched string effect. Recall that the worldsheet correlators for X can be written as¹⁴

$$\mathcal{G}^{\mu\nu}(i\tau_i, i\tau_j) = -\alpha' G^{\mu\nu} \ln \left| \vartheta_1\left(\frac{\tau_{ij}}{2\pi t} \middle| \frac{i}{t}\right) \right|^2 - \frac{i}{2} \Theta^{\mu\nu} \left[\epsilon(\tau_{ij}) - \frac{\tau_{ij}}{\pi t} \right]; \quad (4.1a)$$

$$\mathcal{G}^{\mu\nu}(\pi + i\tau_x, \pi + i\tau_y) = -\alpha' G^{\mu\nu} \ln \left| \vartheta_1\left(\frac{\tau_{xy}}{2\pi t} \middle| \frac{i}{t}\right) \right|^2 + \frac{i}{2} \Theta^{\mu\nu} \left[\epsilon(\tau_{xy}) - \frac{\tau_{xy}}{\pi t} \right]; \quad (4.1b)$$

$$\mathcal{G}^{\mu\nu}(i\tau_i, \pi + i\tau_x) = -\alpha' G^{\mu\nu} \ln \left| \vartheta_4\left(\frac{\tau_{ix}}{2\pi t} \middle| \frac{i}{t}\right) \right|^2 + \frac{(\Theta G \Theta)^{\mu\nu}}{8\pi\alpha' t}. \quad (4.1c)$$

When $B = 0$, in the limit $\alpha' \rightarrow 0$, keeping $\tilde{T} = 2\pi\alpha' t$ and $\tau_p \rightarrow \alpha'\tau_p$ fixed, the above propagators all reduce to those on a circle of length \tilde{T} ,

$$\mathcal{D}^{\mu\nu}(\tau_p, \tau_q) = -G^{\mu\nu} \frac{|\tau_{pq}|(\tilde{T} - |\tau_{pq}|)}{\tilde{T}}. \quad (4.2)$$

However when $\Theta \neq 0$, for nonplanar processes, the worldsheet has a finite length $|\Theta k|$ in the σ -direction, in the limit $\alpha' \rightarrow 0$, and there are nontrivial correlations in the

¹³One might hope to use the methods of [41, 49, 42, 43, 15, 39] to write down a gauge invariant off-shell extension, but we have not succeeded in doing so.

¹⁴Here we take a symmetric form of the propagators with respect to the two boundaries, with $0 \leq \tau_i < 2\pi t$; see appendix B.

worldsheet. In this case, we have,

$$\mathcal{G}^{\mu\nu}(i\tau_i, i\tau_j) = \mathcal{D}^{\mu\nu}(\tau_i, \tau_j) - \frac{i}{2}\Theta^{\mu\nu}\left[\epsilon(\tau_{ij}) - 2\frac{\tau_{ij}}{\tilde{T}}\right]; \quad (4.3a)$$

$$\mathcal{G}^{\mu\nu}(\pi + i\tau_x, \pi + i\tau_y) = \mathcal{D}^{\mu\nu}(\tau_x, \tau_y) + \frac{i}{2}\Theta^{\mu\nu}\left[\epsilon(\tau_{xy}) - 2\frac{\tau_{xy}}{\tilde{T}}\right]; \quad (4.3b)$$

$$\mathcal{G}^{\mu\nu}(i\tau_i, \pi + i\tau_x) = \mathcal{D}^{\mu\nu}(\tau_i, \tau_x) + \frac{(\Theta G \Theta)^{\mu\nu}}{4\tilde{T}}. \quad (4.3c)$$

Thus, the field theory limit remains “stringy”; there is a nontrivial two dimensional worldsheet and there are nontrivial correlations in the worldsheet. After all, the strong coupling limit of this theory is known to be given by a string theory [4, 5] with $\alpha'_{\text{eff}} \sim \Theta$. It is not inconceivable that one might see stringy effects in perturbation theory.¹⁵

It is well-known that it is a subtle issue to extend the first-quantized string theory off-shell. The field variables used in string field theory are normally related to those in the low energy expansion by complicated field redefinitions. Here we may have a similar situation. It would be interesting to perform the noncommutative analog of the computation [52] of the one-loop effective action in field theory, to try to determine, from the field theory point of view, where off-shell gauge invariance breaks down.

For a hint at what the appropriate off-shell variables might be which respect gauge invariance, we compare our computations to those in [37], in which the disk diagram in figure 2, describing the interaction of two photons with a massless closed string, was computed. Our one-loop calculation is related to the amplitude of figure 2 by factorization, so it is not surprising that in both cases $*'$ -products arise. Interestingly, Garousi has interpreted the $*'$ -product via the Seiberg-Witten map [3]. (To conform with the notations of [37] and [3], we will now call our F of previous sections, \hat{F} .)

We now briefly summarize the result of Garousi. The disk amplitude, figure 2, has an expansion in terms of the kinematic invariant $t = -\alpha' p_1 \cdot p_2$ (the dot is with respect to the open string metric). For this amplitude, we can again attempt to write down a gauge invariant off-shell coupling between the massless closed and open string modes, which would reproduce the leading terms in t . However, a closer look at the amplitude reveals a similar problem as in our 1-loop case: such a gauge invariant coupling can apparently not be written in terms of the noncommutative \hat{F} . In [37], it was therefore argued that the appropriate off-shell open string field variables coupling to the closed string modes are commutative F and not the noncommutative \hat{F} of (3.22).

¹⁵S.-J. Rey (private communication) has pointed out that the stretched string is reminiscent of the stringy W -bosons discussed in [51], and that, in particular, the form of the $*'$ -product strongly resembles some stringy structure found in [51].

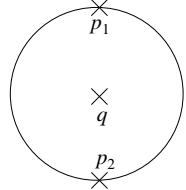


Figure 2: The $*'$ -product also appears in the tree-level contribution to the interaction of a closed string with two open strings.[37]

It is useful to be more explicit. Garousi claims that the lowest order terms in t may be obtained from the following procedure:

1. Start with the Born-Infeld action with commutative F coupled with closed string modes (h below denotes collectively massless closed string modes $\phi, b_{\mu\nu}, h_{\mu\nu}$) and expand to second order in F , *i.e.* (schematically)

$$\mathcal{L}' \sim hF + hFF + \dots \quad (4.4)$$

2. The Seiberg-Witten map [3] gives¹⁶

$$F = \hat{F} + \hat{F} *' \hat{F} + \dots \quad (4.5)$$

where the above expansion is exact in Θ and perturbative in \hat{F} . In the second term $*'$ is precisely what we obtained in (1.8).

3. Substitute (4.5) into (4.4) and at the same time replace all binary products in (4.4) between \hat{F} by $*'$; there is no ambiguity in the order of $*'$ -products since only terms up to second order in field strengths are kept. The resulting Lagrangian may be written

$$\mathcal{L} \sim h\hat{F} + h\hat{F} *' \hat{F} + \dots \quad (4.6)$$

4. The lowest order terms in t for the diagram in figure 2 (higher order terms are related to the exchange of massive open string modes) are exactly reproduced from the scattering amplitude of h and two \hat{F} calculated from the new Lagrangian \mathcal{L} .

Note that the terms in (4.6), that are quadratic in \hat{F} , are not gauge invariant, although the original (commutative) DBI action *is* gauge invariant. Hence it is the commutative fields that are appropriate for describing the process of figure 2.

From the close relation between our one-loop amplitude and the closed string disk amplitudes, one might wonder whether the commutative F is also relevant for writing down an off-shell gauge invariant 1-loop effective action. While this appears natural from string theory, it is certainly counterintuitive from the standpoint of noncommutative field theory; the field theory hardly knows commutative F . We note that in terms of the commutative field strength, which couples to the closed string metric, the expansion we were doing in (3.3) is no longer considered a low-energy expansion. Thus it is also not clear that we can write down an effective action using F either.

¹⁶Equation (4.5) was obtained by integrating the infinitesimal form of the Seiberg-Witten transform along a particular path. It is expected that the path dependence[53] of the Seiberg-Witten map can be absorbed into a field redefinition.

Finally, it would be interesting to check whether it is possible to recover $*'_3$ from the Seiberg-Witten map in the third order term in (4.5). It would also be interesting to determine whether, upon integrating the Seiberg-Witten equation to n^{th} order in the field strength, one obtains n different $*_m$ m -ary operations, $m = 1, \dots, n$, or whether this pattern terminates or converges.

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A. Fermions in a B -field

The fermionic string in the presence of a B -field was discussed in [54, 55, 3, 56, 57]. Here we shall give a self-contained discussion of the fermionic worldsheet fields of the NSR formalism, in the presence of a (constant) B -field and a worldsheet boundary. For concreteness, we take the worldsheet to be the strip, parameterized by $w = \sigma + i\tau$; $\sigma \in [0, \pi]$, $\tau \in \mathbb{R}$ or S^1 .

The bulk action for the fermions ψ^μ and $\tilde{\psi}^\mu$ is

$$S = \frac{1}{4\pi\alpha'} \int d^2w \left\{ (g + B)_{\mu\nu} \psi^\mu \bar{\partial} \psi^\nu + (g - B)_{\mu\nu} \tilde{\psi}^\mu \partial \tilde{\psi}^\nu \right\}. \quad (\text{A.1})$$

The bulk action is invariant under two supersymmetries, under which,

$$\delta X^\mu = -\epsilon \psi^\mu - \bar{\epsilon} \tilde{\psi}^\mu \quad \delta \psi^\mu = \epsilon \partial X^\mu \quad \delta \tilde{\psi}^\mu = \bar{\epsilon} \bar{\partial} X^\mu, \quad (\text{A.2})$$

where ϵ is the complex supersymmetry parameter. When $B = 0$, variation of the action gives

$$\psi^\mu = \pm i \tilde{\psi}^\mu \quad (\text{A.3})$$

at the boundary; this combined with the (Neumann) bosonic boundary condition leads to the preservation of only the supersymmetry for which $\epsilon = \mp i\bar{\epsilon}$. We can choose the

$+$ -sign at one boundary; the choice of sign on the second boundary then gives us the Neveu-Schwarz (NS) or Ramond (R) sectors, for respectively the $-$ or $+$ signs. In the NS sector, the two boundaries preserve different supersymmetries, and so globally, no (worldsheet) supersymmetry is preserved.

For general B , the bosonic action leads to the boundary condition

$$g_{\mu\nu}\partial_n X^\nu - iB_{\mu\nu}\partial_\tau X^\nu|_{\text{bdy}} = 0, \quad (\text{A.4})$$

where ∂_n and ∂_τ are respectively normal and tangential derivatives. The unbroken supersymmetry—namely that for which $\epsilon = \mp i\bar{\epsilon}$ —and equation (A.4), leads to

$$(g + B)_{\mu\nu}\psi^\nu = \pm i(g - B)_{\mu\nu}\tilde{\psi}^\nu, \quad (\text{A.5})$$

which generalizes (A.3) to $B \neq 0$. We have derived equation (A.5) from supersymmetry and not from the action; for a derivation from the action, see [57]. We can define $\tilde{\psi}$ so that the $+$ -sign holds at $\sigma = \pi$. Then we can formally extend the strip to be periodic in σ with periodicity 2π , by setting $\psi^\mu(\sigma, \tau) = i\left(\frac{g-B}{g+B}\right)^\mu{}_\nu\tilde{\psi}^\nu(2\pi - \sigma, \tau)$. The boundary condition at $\sigma = 0$ becomes (anti)periodicity of ψ ; this is just the doubling trick extended to $B \neq 0$.

To summarize, the only change in the manipulation of the fermions when $B \neq 0$ is in the details of the doubling trick. The unbroken supersymmetry is unchanged by the background field, and so the boundary fields at $\sigma = 0, \pi$, $(X^\mu(\tau), \Psi^\mu(\tau))$, which obey

$$\delta X^\mu(\tau) = -\epsilon\Psi^\mu(\tau), \quad \delta\Psi^\mu(\tau) = \epsilon\dot{X}^\mu(\tau), \quad (\text{A.6})$$

are given by

$$\begin{aligned} \Psi^\mu(0, \tau) &= \frac{1}{2}[\psi^\mu(0, \tau) \pm i\tilde{\psi}^\mu(0, \tau)] = \left(\frac{1}{g-B}\right)^\mu{}_\nu\psi^\nu(0, \tau) \\ \Psi^\mu(\pi, \tau) &= \frac{1}{2}[\psi^\mu(\pi, \tau) + i\tilde{\psi}^\mu(\pi, \tau)] = \left(\frac{1}{g-B}\right)^\mu{}_\nu\psi^\nu(\pi, \tau) \end{aligned} \quad (\text{A.7})$$

where, on the rightmost-side of (A.7), ψ is given by that after the doubling trick.

The partition function for the various spin structures is given by [25]

$$\langle 1 \rangle_{+-} = \frac{\vartheta_2^4(0|it)}{\eta(it)^4}, \quad \langle 1 \rangle_{--} = \frac{\vartheta_3^4(0|it)}{\eta(it)^4}, \quad (\text{A.8a})$$

$$\langle 1 \rangle_{-+} = \frac{\vartheta_4^4(0|it)}{\eta(it)^4}, \quad \langle 1 \rangle_{++} = \frac{\vartheta_1^4(0|it)}{\eta(it)^4}, \quad (\text{A.8b})$$

where the subscript $\pm\pm'$ denotes the periodicity in the σ and τ directions respectively. The R (NS) sector is (anti)periodic in σ and antiperiodic in $\tau \sim \tau + 2\pi t$; the corresponding $(-1)^F$ sectors are periodic in τ .

The Green functions in the NS sector for the fermions on the annulus are determined, via the doubling trick, by holomorphy, (anti)periodicity and the OPE $\psi^\mu(w)\psi^\nu(w') \sim \frac{\alpha'}{2} \frac{g^{\mu\nu}}{w-w'}$. The Green functions are thus [58]

$$\langle \psi^\mu(w_1)\psi^\nu(w_2) \rangle_{+-} = \frac{\alpha'}{4\pi} g^{\mu\nu} \frac{\vartheta_2\left(\frac{w_1-w_2}{2\pi}|it\right) \vartheta_1'(0|it)}{\vartheta_1\left(\frac{w_1-w_2}{2\pi}|it\right) \vartheta_2(0|it)}, \quad (\text{A.9a})$$

$$\langle \psi^\mu(w_1)\psi^\nu(w_2) \rangle_{--} = \frac{\alpha'}{4\pi} g^{\mu\nu} \frac{\vartheta_3\left(\frac{w_1-w_2}{2\pi}|it\right) \vartheta_1'(0|it)}{\vartheta_1\left(\frac{w_1-w_2}{2\pi}|it\right) \vartheta_3(0|it)}, \quad (\text{A.9b})$$

$$\langle \psi^\mu(w_1)\psi^\nu(w_2) \rangle_{-+} = \frac{\alpha'}{4\pi} g^{\mu\nu} \frac{\vartheta_4\left(\frac{w_1-w_2}{2\pi}|it\right) \vartheta_1'(0|it)}{\vartheta_1\left(\frac{w_1-w_2}{2\pi}|it\right) \vartheta_4(0|it)}. \quad (\text{A.9c})$$

Equation (A.7) immediately implies that correlation functions of the boundary fermions Ψ are given by (A.9), with the closed string metric $g^{\mu\nu}$ replaced with the open string metric $G^{\mu\nu} = \left(\frac{1}{g+B} g \frac{1}{g-B}\right)^{\mu\nu}$.

For general periodicity $\psi^\mu(w + 2n\pi + 2m\pi it) = e^{2\pi i[n(\alpha-1/2)+m(\beta-1/2)]} \psi^\mu(w)$, we can write [58]

$$\langle \psi^\mu(w_1)\psi^\nu(w_2) \rangle = \frac{\alpha'}{4\pi} g^{\mu\nu} \frac{\vartheta\left[\frac{\alpha}{\beta}\right]\left(\frac{w_1-w_2}{2\pi}|it\right) \vartheta'\left[\frac{1/2}{1/2}\right](0|it)}{\vartheta\left[\frac{1/2}{1/2}\right]\left(\frac{w_1-w_2}{2\pi}|it\right) \vartheta\left[\frac{\alpha}{\beta}\right](0|it)}, \quad (\text{A.9'})$$

and the reader who is not confused by the observation that the doubly periodic case of $\alpha = \beta = 1/2$ in equation (A.9') is everywhere singular, can skip the rest of this paragraph. Of course, for this spin structure, we cannot write down a Green function with the (allegedly) desired properties, because of the theorem [26, p. 431] that there are no doubly periodic meromorphic functions with a single simple pole. Physically, this is the result of the existence of a (constant) zero mode on the doubly periodic torus. Thus the doubly periodic spin structure does not contribute to correlation functions with fewer than 10 fermions. Since the correlation functions that we consider have no more than 8 fermions, we can simply ignore the doubly periodic spin structure in this paper.

B. A Note on the Boundary Green Function

In this section we give a derivation of the boundary propagators on the annulus in the presence of a B -field, since there is some confusion in the literature as to what the correct Green function is. Our discussion will be based on the results in [31].

The goal is to find a solution of

$$\partial_1 \bar{\partial}_1 \mathcal{G}^{\mu\nu}(w_1, w_2) = -\pi\alpha' g^{\mu\nu} \delta^{(2)}(w_1 - w_2) + \frac{\alpha'}{4\pi t} g^{\mu\nu}, \quad (\text{B.1a})$$

$$\{(g + B)_{\mu\lambda} \partial_1 \mathcal{G}^{\lambda\nu}(w_1, w_2) + (g - B)_{\mu\lambda} \bar{\partial}_1 \mathcal{G}^{\lambda\nu}(w_1, w_2)\} \Big|_{\sigma_1=0, \pi} = 0, \quad (\text{B.1b})$$

$$\mathcal{G}^{\mu\nu}(w_1 + 2\pi it, w_2) = \mathcal{G}^{\mu\nu}(w_1, w_2) = \mathcal{G}^{\mu\nu}(w_1, w_2 + 2\pi it). \quad (\text{B.1c})$$

Equation (B.1a) is the equation of motion for X^μ and eq. (B.1b) is the boundary condition for $X^\mu(w_1)$, as modified by the B -field in the now-familiar way. Symmetry, $\mathcal{G}^{\mu\nu}(w_1, w_2) = \mathcal{G}^{\nu\mu}(w_2, w_1)$ enforces the boundary condition on $X^\nu(w_2)$, so we do not need to impose that separately. The last term in equation (B.1a) is a background charge without which (B.1a) would be inconsistent with (B.1b) and Gauss' law [25]. Finally, the periodicity condition (B.1c) enforces singlevaluedness of the Green function on the annulus.

Chaudhury and Novac [31] solve equations (B.1) with¹⁷

$$\begin{aligned} \mathcal{G}^{\mu\nu}(w_1, w_2) = & -\frac{\alpha'}{2} G^{\mu\nu} \left\{ \ln \left| \frac{\vartheta_1\left(\frac{w_1-w_2}{2\pi it} \mid \frac{i}{t}\right) \vartheta_1\left(\frac{w_1+\bar{w}_2}{2\pi it} \mid \frac{i}{t}\right)}{\eta^2\left(\frac{i}{t}\right)} \right|^2 - \frac{(\text{Re } w_1)^2 + (\text{Re } w_2)^2}{\pi t} \right\} \\ & - \frac{\Theta^{\mu\nu}}{2\pi} \left\{ \log \vartheta_1\left(\frac{w_1 + \bar{w}_2}{2\pi it} \mid \frac{i}{t}\right) - \log \vartheta_1\left(\frac{\bar{w}_1 + w_2}{2\pi it} \mid \frac{i}{t}\right) - \frac{i}{t} \text{Im}(w_1 - w_2) \right\} \\ & + \frac{(\Theta G \Theta)^{\mu\nu}}{8\pi^2 \alpha'} \left\{ \ln \left| \frac{\vartheta_1\left(\frac{w_1-w_2}{2\pi it} \mid \frac{i}{t}\right)}{\vartheta_1\left(\frac{w_1+\bar{w}_2}{2\pi it} \mid \frac{i}{t}\right)} \right|^2 - \frac{[\text{Re } w_1 - \pi]^2 + [\text{Re } w_2 - \pi]^2}{\pi t} + \frac{2\pi}{t} \right\}. \quad (\text{B.2}) \end{aligned}$$

However, equation (B.2) is not complete without specifying a branch for the logarithm for which the Green function is continuous. In equation (B.2) the dangerous logarithms reside in the function multiplying $\Theta^{\mu\nu}$. As the arguments of the logarithms cross the branch cut, there are possible discontinuities in the Green function, which will result in extra, δ' -function sources on the right hand side of equation (B.1a). Thus, as it is written here, the Green function does not actually obey the equation of motion (B.1a).

It would be desirable to choose a branch cut that is never crossed in the range of the variables of the Green function; then the Green function would automatically be continuous. An example of this is the Green function for the upper-half plane with the branch cut along the positive real axis.¹⁸ However, for the annulus, the nature

¹⁷A similar formula in ref. [30] does not have the linear terms in each of the second and third lines. These terms are related by the boundary condition (B.1b) and do not affect the equation of motion (B.1a). Our careful analysis of periodicity (B.1c) shows that these terms are essential.

¹⁸The boundary is not considered part of the worldsheet; in particular, we define the boundary Green function by taking the limit from the interior.

of the θ -functions is such that there is no choice of branch cut that is never crossed. Thus the function multiplying $\Theta^{\mu\nu}$ in the second line of equation (B.2) is discontinuous, and will produce additional δ' -functions proportional to $\Theta^{\mu\nu}$ on the right hand side of the equation of motion (B.1a). The remedy is to add a function to equation (B.2) to explicitly cancel the unphysical discontinuities (of course, we will ensure that this function affects neither the equation of motion nor the boundary condition).

The simplest choice of branch cut appears to be along the negative real axis. We choose $-\pi < \text{Im } \log z \leq \pi$ and denote this by writing $\text{Log}_{(-\pi, \pi]}$. It can be seen from the expression

$$\vartheta_1\left(\frac{\sigma + i\tau}{2\pi it} \mid \frac{i}{t}\right) = 2 \sum_{n=0}^{\infty} (-1)^n e^{-\frac{\pi}{t}(n+\frac{1}{2})^2} \left[\sin\left((n + \frac{1}{2})\frac{\tau}{t}\right) \cosh\left((n + \frac{1}{2})\frac{\sigma}{t}\right) - i \cos\left((n + \frac{1}{2})\frac{\tau}{t}\right) \sinh\left((n + \frac{1}{2})\frac{\sigma}{t}\right) \right], \quad (\text{B.3})$$

that this branch cut is crossed, in a counterclockwise direction, by $\vartheta_1\left(\frac{w_1 + \bar{w}_2}{2\pi it} \mid \frac{i}{t}\right)$ when τ_{12} increases through $\tau_{12} = (4n - 1)\pi t$ for $n \in \mathbb{Z}$. (Recall our notation is $w_i = \sigma_i + i\tau_i$ where $\sigma_i \in (0, \pi)$ and $\tau_i \in \mathbb{R}$; thus $0 < \sigma_1 + \sigma_2 < 2\pi$.) Similarly, $\vartheta_1\left(\frac{\bar{w}_1 + w_2}{2\pi it} \mid \frac{i}{t}\right)$ crosses the branch cut in a clockwise direction as τ_{12} increases through $\tau_{12} = (4n + 1)\pi t$. We believe that these are the only crossings. Thus the discontinuities of the function multiplying $\Theta^{\mu\nu}$ in the second line of equation (B.2),

$$\text{Log}_{(-\pi, \pi]} \vartheta_1\left(\frac{\sigma + i\tau}{2\pi it}\right) - \text{Log}_{(-\pi, \pi]} \vartheta_1\left(\frac{\sigma - i\tau}{2\pi it}\right), \quad (\text{B.4})$$

can be mimicked by a function $2\pi i f(\tau)$ with f defined by

$$f(\tau) = -\left[\frac{\tau}{2\pi t}\right], \quad (\text{B.5})$$

where $[x]$ denotes the closest integer to x .¹⁹

Thus, the difference of equation (B.4) and $2\pi i f(\tau)$ is continuous across the branch cut, but since $2\pi i f(\tau)$ is otherwise constant, it preserves the (differential) properties of

¹⁹Continuity of equation (B.6) further sets $[\frac{1}{2} + 2n] = 2n$ and $[\frac{3}{2} + 2n] = 2(n + 1)$ for $n \in \mathbb{Z}$.

equation (B.4) away from the branch cut. So, by replacing equation (B.2) with

$$\begin{aligned}
\mathcal{G}^{\mu\nu}(w_1, w_2) = & -\frac{\alpha'}{2} G^{\mu\nu} \left\{ \ln \left| \frac{\vartheta_1\left(\frac{w_1-w_2}{2\pi it} \mid \frac{i}{t}\right) \vartheta_1\left(\frac{w_1+\bar{w}_2}{2\pi it} \mid \frac{i}{t}\right)}{\eta^2\left(\frac{i}{t}\right)} \right|^2 - \frac{(\text{Re } w_1)^2 + (\text{Re } w_2)^2}{\pi t} \right\} \\
& - \frac{\Theta^{\mu\nu}}{2\pi} \left\{ \text{Log}_{(-\pi, \pi]} \vartheta_1\left(\frac{w_1+\bar{w}_2}{2\pi it} \mid \frac{i}{t}\right) - \text{Log}_{(-\pi, \pi]} \vartheta_1\left(\frac{\bar{w}_1+w_2}{2\pi it} \mid \frac{i}{t}\right) \right. \\
& \quad \left. - \frac{i}{t} \text{Im}(w_1 - w_2) - 2\pi i f(\text{Im}(w_1 - w_2)) \right\} \\
& + \frac{(\Theta G \Theta)^{\mu\nu}}{8\pi^2 \alpha'} \left\{ \ln \left| \frac{\vartheta_1\left(\frac{w_1-w_2}{2\pi it} \mid \frac{i}{t}\right)}{\vartheta_1\left(\frac{w_1+\bar{w}_2}{2\pi it} \mid \frac{i}{t}\right)} \right|^2 - \frac{[\text{Re } w_1 - \pi]^2 + [\text{Re } w_2 - \pi]^2}{\pi t} + \frac{2\pi}{t} \right\}, \quad (\text{B.6})
\end{aligned}$$

we now satisfy equations (B.1), even across the branch cut.

Having modified the antisymmetric part of (B.2) by some step functions, we should check that it indeed obeys the periodicity condition (B.1c). (The periodicity of the symmetric part of (B.6) is obvious.) It is enough to check periodicity under $w_1 \rightarrow w_1 + 2\pi it$, since the Green function depends on the imaginary parts of w_1 and w_2 only as $\text{Im}(w_1 - w_2)$; this is translational invariance along the cylinder. Under $w_1 \rightarrow w_1 + 2\pi t$, the ϑ_1 -function changes sign; this is the standard antiperiodicity of ϑ_1 . If, for some $n \in \mathbb{Z}$, $(4n-1)\pi t < \text{Im}(w_1 - w_2) < (4n+1)\pi t$, then the first Log gives an additional πi (the translation of w_1 yields a counterclockwise rotation of the ϑ -function which does not cross the branch cut). The second Log is essentially a reflection across the imaginary axis, and thus is a clockwise rotation that does cross the branch cut; this then also gives πi and the difference vanishes. Similarly, the difference of the Log ϑ_1 s also cancels in the complementary region $(4n+1)\pi t < \text{Im}(w_1 - w_2) < (4n+3)\pi t$.²⁰ Finally, it is clear that $2\pi i f(\text{Im}(w_1 - w_2))$ jumps by $-2\pi i$ under a shift of w_1 by $2\pi t$,²¹ which cancels the shift in $\text{Im}(w_1 - w_2)$. Thus the Green function (B.6) is periodic. Note that the explicit inclusion of $\text{Im}(w_1 - w_2)$ was crucial (cf. footnote 17).

Equations (4.1) then follow upon taking the limit that w_1, w_2 are on the boundary, in the region $0 \leq \tau_1, \tau_2 < 2\pi t$. Outside this region, the actual expression is more complicated, but it follows from periodicity.

$\sigma_1 = \sigma_2 = \mathbf{0}$: First consider $-\pi t < \tau_{12} < \pi t$. In this region, one can show that

$$\vartheta_1\left(\frac{\delta + i\tau_{12}}{2\pi it} \mid \frac{i}{t}\right) \sim -i\delta + \epsilon(\tau_{12}). \quad (\text{B.7})$$

²⁰This leaves $\text{Im}(w_1 - w_2) = (2n+1)\pi t$. This is more involved and the specification of footnote 19 is vital.

²¹Again the analysis is more involved when $\text{Im}(w_1 - w_2) = (2n+1)\pi t$.

Therefore,

$$\mathcal{G}^{\mu\nu}(i\tau_1, i\tau_2) = -\alpha' G^{\mu\nu} \ln \left| \frac{\vartheta_1\left(\frac{\tau_{12}}{2\pi t} \mid \frac{i}{t}\right)}{\eta\left(\frac{i}{t}\right)} \right|^2 + \frac{i}{2} \Theta^{\mu\nu} \left[\frac{\tau_{12}}{\pi t} - \epsilon(\tau_{12}) \right], \quad -\pi t < \tau_{12} < \pi t, \quad (\text{B.8})$$

Periodicity extends this result to $-2\pi t < \tau_{12} < 2\pi t$; remarkably, the form is the same. (However, outside this larger region, the form changes.) This reproduces equation (4.1a).²²

$\sigma_1 = \sigma_2 = \pi$: Quasi-periodicity of the ϑ -function (see e.g. [26]) gives

$$\vartheta_1\left(\frac{\tau_{12} + i\delta}{2\pi t} - \frac{i}{t} \mid \frac{i}{t}\right) = -e^{\frac{\pi}{t} - \frac{\delta}{t} + i\frac{\tau_{12}}{t}} \vartheta_1\left(\frac{\tau_{12} + i\delta}{2\pi t} \mid \frac{i}{t}\right). \quad (\text{B.9})$$

Thus, for $-\pi t < \tau_{12} < \pi t$, and using equation (B.7),

$$\begin{aligned} \text{Log}_{(-\pi, \pi]} \vartheta_1\left(\frac{2\pi - \delta + i\tau_{12}}{2\pi t} \mid \frac{i}{t}\right) - \text{Log}_{(-\pi, \pi]} \vartheta_1\left(\frac{2\pi - \delta - i\tau_{12}}{2\pi t} \mid \frac{i}{t}\right) \\ = -\pi i \epsilon(\tau_{12}) + 2i \frac{\tau_{12}}{t}. \end{aligned} \quad (\text{B.10})$$

Hence,

$$\mathcal{G}^{\mu\nu}(\pi + i\tau_1, \pi + i\tau_2) = -\alpha' G^{\mu\nu} \ln \left| \frac{\vartheta_1\left(\frac{\tau_{12}}{2\pi t} \mid \frac{i}{t}\right)}{\eta\left(\frac{i}{t}\right)} \right|^2 - \frac{i}{2} \Theta^{\mu\nu} \left[\frac{\tau_{12}}{\pi t} - \epsilon(\tau_{12}) \right], \quad -\pi t < \tau_{12} < \pi t; \quad (\text{B.11})$$

again, this has the same form when extended to $-2\pi t < \tau_{12} < 2\pi t$. So we have equation (4.1b).

$\sigma_1 = \mathbf{0}, \sigma_2 = \pi$: Using

$$\vartheta_1\left(\frac{w_1 + \bar{w}_2}{2\pi it} \mid \frac{i}{t}\right) = \vartheta_1\left(\frac{\tau_{12} - i\delta}{2\pi t} - \frac{i}{2t} \mid \frac{i}{t}\right) = -ie^{\frac{\pi}{4t} + i\frac{\tau_{12}}{2t}} \vartheta_4\left(\frac{\tau_{12} - i\delta}{2\pi t} \mid \frac{i}{t}\right) \quad (\text{B.12})$$

and the fact that $\vartheta_4\left(\frac{\tau_{12}}{2\pi t} \mid \frac{i}{t}\right) > 0$ for all (real) τ_{12} , we immediately find

$$\mathcal{G}^{\mu\nu}(i\tau_1, \pi + i\tau_2) = -\alpha' G^{\mu\nu} \ln \left| \frac{\vartheta_4\left(\frac{\tau_{12}}{2\pi t} \mid \frac{i}{t}\right)}{\eta\left(\frac{i}{t}\right)} \right|^2 + \frac{(\Theta G \Theta)^{\mu\nu}}{8\pi\alpha' t}. \quad (\text{B.13})$$

In particular, the antisymmetric part vanishes, so periodicity is trivial. This is equation (4.1c). Note that $\sigma_1 = \pi, \sigma_2 = 0$ is exactly the same since $\text{Re}(w_1 + \bar{w}_2) = \sigma_1 + \sigma_2$.

²²We have dropped an overall $2\alpha' G^{\mu\nu} \log \eta\left(\frac{i}{t}\right)$ from each of equations (4.1); this is just a (t -dependent) constant.

Finally, one can check, though one must be careful to keep certain self-contractions, that the two-point function for two gauge bosons in the bosonic string correctly reproduces the field theory one-loop contribution to the two-point function. This was the evidence that caused [29] to assert the correctness of their boundary Green function, even though that one turns out to not be periodic.

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